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STRONGLY ALMOST DISJOINT FUNCTIONS

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ABSTRACT

Two total functions f, g from ω_1 to ω are called strongly almost disjoint if $\{\alpha < \omega_1: f(\alpha) = g(\alpha)\}$ is finite. We show that it is consistent with ZFC to have families of pairwise strongly almost disjoint functions of arbitrary prescribed size.

Introduction

The properties of the structures $\mathcal{P}(\omega_1)$ modulo countable, ${}^{\omega_1}\omega_1, <$ modulo countable etc. have been fairly well investigated [J, JS]. The same cannot be said about the structures of the form $\mathcal{P}(\omega_1)$ modulo finite, ${}^{\omega_1}\omega_1, <$ modulo finite and the like. Many apparently hard open problems persist in this area. Here is one prominent example:

QUESTION 1 (Hajnal): *How long chains can there be in ${}^{\omega_1}\omega_1, <$ modulo finite?*

We embark on an investigation of a lesser problem. We define

Definition 2: Functions f, g from ω_1 to ω are called **strongly almost disjoint** if the set $\{\alpha \in \omega_1: f(\alpha) = g(\alpha)\}$ is finite. A family $\{f_i: i \in I\} \subset {}^{\omega_1}\omega$ is called strongly almost disjoint if it consists of pairwise strongly almost disjoint functions.

and ask about possible sizes of strongly almost disjoint families. Notice that if there is a chain in ${}^{\omega_1}\omega_1, <$ modulo finite of length $\kappa + 1$ then there is a strongly almost disjoint family of size κ. Therefore this problem represents an intermediate step towards a solution of Question 1. We prove

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THEOREM 3: *Assume that the Continuum Hypothesis holds and choose a cardinal κ. Then there is a cardinal-preserving forcing P such that $P \Vdash$ "there is a strongly almost disjoint family of size κ".*

Not much has been known about the related problems before. An old result of Baumgartner's [B] says that for any given κ, consistently there is a family of size κ of cofinal subsets of ω_1 with pairwise finite intersections. Note that a strongly almost disjoint family in our sense is just a family of modulo finite disjoint subsets of $\omega_1 \times \omega$ with some special properties. The particular cases of $\kappa = \aleph_1, \aleph_2$ of Theorem 3 have been known, too:

CLAIM 4 (Folk?): *There is a strongly almost disjoint family of size $\aleph_1$.*

Proof: For every $\alpha \in \omega_1$ choose a bijection $h_\alpha: \alpha \to \omega$. We construct a sequence $\langle f_\gamma: \gamma \in \omega_1 \rangle \subset {}^{\omega_1}\omega$ so that the following holds:

(1) for every $\alpha \geq \gamma$ we have $f_\gamma(\alpha) = h_\alpha(\gamma)$,
(2) for $\delta \neq \gamma$ the set $\{\alpha \in \omega_1: f_\delta(\alpha) = f_\gamma(\alpha)\}$ is finite.

Assume that functions $\langle f_\gamma: \gamma \in \delta \rangle$ have been constructed for $\delta \in \omega_1$. Since this is a countable family of functions, we can find a function $g: \delta \to \omega$ such that for every $\gamma \in \delta$ the set $\{\alpha \in \delta: f_\gamma(\alpha) = g(\alpha)\}$ is finite. Then f_δ is defined by $f_\delta \restriction \delta = g$ and $f_\delta(\alpha) = h_\alpha(\delta)$ for every $\alpha \in \omega_1 \setminus \delta$. Obviously, the family $\{f_\gamma: \gamma \in \omega_1\} \subset {}^{\omega_1}\omega$ will be strongly almost disjoint. ∎

THEOREM 5 ([T2]): *Negation of Chang's conjecture is equivalent to the existence of a c.c.c. poset P forcing a strongly almost disjoint family of size $\aleph_2$.*

Thus our problem, unlike Baumgartner's, is connected to the large cardinal properties of the cardinals concerned. Our proof of Theorem 3 uses the side condition method of [T1] and is of independent interest.

Our notation is standard as set forth in [J]. In particular, H_κ denotes the collection of sets hereditarily of cardinality κ and the sign $\prec$ is reserved for the relation "an elementary submodel of". In a forcing notion, we write $p \leq q$ to mean that p is more informative than q. The presented result is taken from the first chapter of the author's Ph.D. thesis [Z]. The author wants to thank Professor Balcar for asking the question about the validity of Theorem 3.

1. The side condition method

First, we review the basic definitions and facts about countable submodels of large structures as relevant to the side condition method.

Let κ be an uncountable regular cardinal and fix $\ll$, a well-ordering of H_κ. Also, choose one distinguished element Δ of H_κ.

Definition 7: We say that $\mathfrak{m}$ is a matrix of models if the following conditions are satisfied:

(D7.1) $\mathfrak{m}$ is a function, $\mathrm{dom}(\mathfrak{m}) \in [\omega_1]^{<\aleph_0}$ and for each $\alpha \in \mathrm{dom}(\mathfrak{m})$ the value $\mathfrak{m}(\alpha)$ is a finite set of isomorphic countable submodels of $\langle H_\kappa, \in, \ll, \Delta\rangle$,

(D7.2) for each $\alpha < \beta$ both in $\mathrm{dom}(\mathfrak{m})$ we have $\forall N \in \mathfrak{m}(\alpha)\ \exists M \in \mathfrak{m}(\beta)\ N \in M$,

(D7.3) for each $\alpha < \beta$ both in $\mathrm{dom}(\mathfrak{m})$ we have $\forall M \in \mathfrak{m}(\beta)\ \exists N \in \mathfrak{m}(\alpha)\ N \in M$.

We consider the set $\mathfrak{M}$ of all matrices of models to be ordered by $\geq$, the reverse coordinatewise extension. That is, $\mathfrak{n} \geq \mathfrak{m}$ if $\mathrm{dom}(\mathfrak{n}) \subset \mathrm{dom}(\mathfrak{m})$ and for each $\alpha \in \mathrm{dom}(\mathfrak{n})$ we have $\mathfrak{n}(\alpha) \subset \mathfrak{m}(\alpha)$. ∎

The poset $\mathfrak{M}$ is a subset of H_κ and it is not necessarily separative. Its definition has three parameters: the cardinal κ, the well-ordering $\ll$ and the distinguished element Δ. Below, it will be used as a kind of reflection tool, ensuring properness of certain forcings.

It is helpful here to observe three things. First, if N_0, N_1 are isomorphic submodels of $\langle H_\kappa, \in, \ll, \Delta\rangle$, then there is exactly one isomorphism $i\colon N_0 \to N_1$, namely the unique bijection of the two respecting the order $\ll$. Moreover $i \restriction N_0 \cap \omega_1 = \mathrm{id}$ and necessarily $N_0 \cap \omega_1 = N_1 \cap \omega_1$. Second, if $N \in M$ are countable submodels of $\langle H_\kappa, \in, \ll, \Delta\rangle$ then $N \subset M$; this holds because of the countability of N and the elementarity of M. It follows that the relation $\in$ is transitive on the set of countable submodels of $\langle H_\kappa, \in, \ll\rangle$. Third, if $N \in M, \overline{M}$ are three countable elementary submodels of $\langle H_\kappa, \in, \ll, \Delta\rangle$ and $i\colon M \to \overline{M}$ is an isomorphism, then $i(N) \in \overline{M}$ is another elementary submodel isomorphic to N via $i \restriction N$.

We show now that $\langle \mathfrak{M}, \geq\rangle$ is projected in a certain precise sense to countable submodels of $\langle H_\kappa, \in, \ll, \Delta\rangle$, see Lemma 9. This results in properness of $\mathfrak{M}$ as a forcing notion.

Definition 8: Let $M \prec \langle H_\kappa, \in, \ll, \Delta\rangle$ be a countable model and let $\mathfrak{m} \in \mathfrak{M}$ be such that $M \in \mathfrak{m}(M \cap \omega_1)$. Then we define the following notions:

(D8.1) $\mathrm{pr}_M(\mathfrak{m})$, the projection of $\mathfrak{m}$ into $M \cap \mathfrak{M}$. This is the function defined by dom($\mathfrak{n}$) = dom($\mathfrak{m}$) $\cap M$ and $N \in \mathfrak{n}(\alpha)$ iff there are models $N = N_0 \in N_1 \in \cdots \in N_k = M$ such that $N_i \in \mathfrak{m}(\alpha_i)$, where $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_k = M \cap \omega_1$ is an increasing list of all ordinals in dom($\mathfrak{m}$) between α and $M \cap \omega_1$.

(D8.2) A system $\mathfrak{m}$ is said to be M-full if for each $\alpha \in M \cap \mathrm{dom}(\mathfrak{m})$ and each $N \in \mathfrak{m}(\alpha) \setminus \mathrm{pr}_M(\mathfrak{m})(\alpha)$ there is $\overline{M} \in \mathfrak{m}(M \cap \omega_1)$ such that $N \in \overline{M}$ and $i(N) \in \mathrm{pr}_M(\mathfrak{m})(\alpha)$, where $i\colon \overline{M} \to M$ is the unique isomorphism of $\overline{M}$ and M.

The rationale behind this definition is that $\mathrm{pr}_M(\mathfrak{m})$ should be an element of $M \cap \mathfrak{M}$ containing all the information about $\mathfrak{m}$ understandable from within M. The matrix $\mathfrak{m}$ is M-full if $\mathrm{pr}_M(\mathfrak{m})$ works as the standard regular subposet projection of $\mathfrak{M}$ into $M \cap \mathfrak{M}$. These intuitions are expressed in the following easy lemma.

LEMMA 9: *Let $M \prec \langle H_\kappa, \in, \ll, \Delta\rangle$ be a countable model and let $\mathfrak{m} \in \mathfrak{M}$ be such that $M \in \mathfrak{m}(M \cap \omega_1)$. Then*

(L9.1) $\mathrm{pr}_M(\mathfrak{m}) \in M \cap \mathfrak{M}$.

(L9.2) *There is $\mathfrak{n} \le \mathfrak{m}$ such that $\mathfrak{n}$ is M-full.*

(L9.3) *Assume that $\mathfrak{m}$ is M-full. For every $\mathfrak{n} \in M \cap \mathfrak{M}$, if $\mathfrak{n} \le \mathrm{pr}_M(\mathfrak{m})$ then $\mathfrak{n}$ is compatible with $\mathfrak{m}$.*

2. The Proof of Theorem 3

Let the Continuum Hypothesis hold and choose an arbitrary cardinal $\kappa \ge \aleph_2$. The first rather naïve try to obtain the needed forcing P would be a "finite conditions" construction: we could set $P = \{\langle x, y, z\rangle\colon x \in [\omega_1]^{<\aleph_0}, y \in [\kappa]^{<\aleph_0}, z$ is a function with dom$(z) = y$ and for $\gamma \in y$ the value $z(\gamma)$ is a function in ${}^x\omega\}$, ordered by $\langle x_0, y_0, z_0\rangle \ge \langle x_1, y_1, z_1\rangle$ if $x_0 \subset x_1, y_0 \subset y_1$, for every $\gamma \in y_0$ we have $z_0(\gamma) = z_1(\gamma) \restriction x_0$ and for every $\alpha \in x_1 \setminus x_0$ and every $\gamma \ne \delta$ both in y_0 we have $z_1(\gamma)(\alpha) \ne z_1(\delta)(\alpha)$. The idea is that for $\gamma \in y$, the function $z(\gamma)$ is a finite approximation to the future function $f_\gamma \in {}^{\omega_1}\omega$; and $\{f_\gamma\colon \gamma \in \kappa\}$ will be the needed family. However, this forcing collapses $\aleph_1$. To alleviate this problem, we add to the conditions $\langle x, y, z\rangle$ a control device, a matrix of models, and we restrict the possibilities of extending the conditions in order to obtain master conditions for the models in the matrix.

Fix a well-ordering $\ll$ of H_κ and let $\Delta = 0$; below the set $\mathfrak{M}$ is calculated from this data.

Definition 10: P is the set of all p's such that

(D10.1) $p = \langle x^p, y^p, z^p, \mathfrak{m}^p \rangle$, where if no confusion is possible, we drop the superscript p,

(D10.2) $x \in [\omega_1]^{<\aleph_0}, y \in [\kappa]^{<\aleph_0}, z$ is a function with $\mathrm{dom}(z) = y$ such that for $\gamma \in y$ we have $z(\gamma) \in {}^x\omega$,

(D10.3) $\mathfrak{m}$ is a matrix of models ($\mathfrak{m} \in \mathfrak{M}$).

The order is defined by $p \geq q$ if

(D10.4) $x^p \subset x^q, y^p \subset y^q, \forall \gamma \in y^p\ z^p(\gamma) \subset z^q(\gamma)$ and $\mathfrak{m}^p \geq \mathfrak{m}^q$,

(D10.5) for each $\gamma \neq \delta$ both in y^p and each $\beta \in x^q \setminus x^p$ we have $z^q(\gamma)(\beta) \neq z^q(\delta)(\beta)$,

(D10.6) for each $\gamma \neq \delta$ both in y^q, for each $\alpha \in \mathrm{dom}(\mathfrak{m}^p)$, for each $N \in \mathfrak{m}^p(\alpha)$ such that $\gamma, \delta \in N$ and for each $\beta \in x^q \setminus \alpha$, **if** at least one of $z^p(\gamma)(\beta), z^p(\delta)(\beta)$ is undefined **then** $z^q(\gamma)(\beta) \neq z^q(\delta)(\beta)$.

Explanation: In a condition p, the function $z^p(\gamma)$, for $\gamma \in y^p$, is a finite piece of the future function $f_\gamma: \omega_1 \to \omega$. If we put p into a generic filter, we make the following promises:

(1) For $\gamma \neq \delta$ both in y^p we promise that f_γ, f_δ will have different values everywhere on $\omega_1 \setminus x^p$.

(2) For $\gamma \neq \delta$ both in some $N \in \mathfrak{m}^p(\alpha)$ for some α, we promise that f_γ, f_δ will have different values everywhere above α, save perhaps for the values on $x^p \setminus \alpha$ (in the case when both γ, δ are in y^p).

The model part of the conditions ensures that the forcing will be sufficiently mild.

It is immediate that $\langle P, \leq \rangle$ is a partially ordered set. The following two lemmas are of utmost importance. Their proofs are very typical for this so-called "side condition method".

LEMMA 11 (CH): *P has $\aleph_2$-chain condition.*

Proof: Let $\langle p_\nu : \nu \in \omega_2 \rangle$ be a putative antichain, $p_\nu = \langle x^\nu, y^\nu, z^\nu, \mathfrak{m}^\nu \rangle$. Without loss of generality (using pigeonhole principle and a Δ-system lemma) we may assume that there are $x \in [\omega_1]^{<\aleph_0}, y \in [\kappa]^{<\aleph_0}$ and a function z with domain y such that $\forall \nu \in \omega_2\ x^\nu = x$, the sets y^ν form a Δ-system with root y and $\forall \nu \in \omega_2\ z^\nu \restriction y = z$. For each $\xi \in \omega_2$ we pick a countable model $M_\xi \prec \langle H_\kappa, \in, \ll, \langle p_\nu : \nu \in \omega_2 \rangle, x, y, z, \xi \rangle$.

CLAIM 12: *There are ordinals $\xi < \zeta < \omega_2$ and i: $\langle M_\xi, \in, \ll, \xi\rangle \to \langle M_\zeta, \in, \ll, \zeta\rangle$, an isomorphism equal to identity on $M_\xi \cap M_\zeta$.*

Proof: Use CH and a Δ-system argument to find a countable set S and a set $A \subset \omega_2$ of full cardinality such that $\langle M_\xi: \xi \in A\rangle$ form a Δ-system with root S. Let $S = s_0, s_1, \ldots, s_n, \ldots$, $n \in \omega$ be a one-to-one list of elements of S. Expand each $M_\xi, \xi \in A$ to $\langle M_\xi, \in, \ll, \langle p_\nu: \nu \in \omega_2\rangle, x, y, z, \xi, s_n: n \in \omega\rangle$. There are only $\mathfrak{c} = \aleph_1$ many isomorphism types of these, so one can find $\xi \in \zeta$ both in A and an isomorphism i of the expanded structures on M_ξ, M_ζ. Since $M_\xi \cap M_\zeta = S$ and $i \restriction S = \mathrm{id}$, we have exemplified the statement of the Claim with ξ, ζ, i. ∎

We fix $\xi \in \zeta$ as in Claim 12 and define p by $x^p = x, y^p = y^\xi \cup y^\zeta, z^p = z^\xi \cup z^\zeta$. Furthermore $\mathfrak{m}^p$ will be a coordinatewise union of $\mathfrak{m}^\xi$ and $\mathfrak{m}^\zeta$.

CLAIM 13: *$p \in P, p^\xi \geq p, p^\zeta \geq p$.*

Proof: It is immediate that $p \in P$. Notice that the matrices $\mathfrak{m}^\xi, \mathfrak{m}^\zeta$ have the same domain.

Next we want to show that $p^\xi \geq p$. (The proof for p^ζ is parallel.) (D10.4) and (D10.5) are immediate. In (D10.6), if both ordinals γ and δ are in y^ξ then the "if" clause is false and so the whole formula is true. So the only nontrivial case is when at least one of γ, δ does not belong to y^ξ, let us say $\gamma \notin y^\xi$. But then, $\gamma \in y^\zeta \setminus y$ and so necessarily $\gamma \notin M_\xi$. (*Proof.* Assume for contradiction that $\gamma \in M_\xi$. Then, as ζ is the unique ordinal in ω_2 for which $\gamma \in y^\zeta$, we have $\zeta \in M_\xi$. So $\zeta \in M_\xi \cap M_\zeta$ and consequently $i(\zeta) = \zeta$. Since also $i(\xi) = \zeta$ and $\xi \neq \zeta$, this contradicts i being an isomorphism.) Therefore for no $\alpha \in \mathrm{dom}(\mathfrak{m}^\xi)$ and no $N \in \mathfrak{m}^\xi(\alpha)$ we have $\gamma \in N$. This is because such an N should be an element, and therefore subset, of M_ξ. So (D10.6) is vacuously true for the pair γ, δ, as the third universal quantifier is vacuous. ∎

Claim 13 gives a contradiction with our assumption about p_ν's forming an antichain. ∎

LEMMA 14: *P is proper.*

Proof: Choose $p_0 \in P$ and a countable submodel $M \prec H_\lambda$ with $\kappa, p_0, \ll$ in M. We shall produce a master condition $p_1 \leq p_0$ for the model M. We set $x^{p_1} = x^{p_0}, y^{p_1} = y^{p_0}, z^{p_1} = z^{p_0}$ and let $\mathfrak{m}^{p_1} = \mathfrak{m}^{p_0} \cup \langle M \cap \omega_1, \{M \cap H_\kappa\}\rangle$.

CLAIM 15: $p_1 \in P, p_1 \leq p_0$.

We have to verify that the condition p_1 is master for M. That is, for every maximal antichain A of P with $A \in M$, the set $A \cap M$ should be predense below p_1. To prove this, fix a maximal antichain $A \in M$ and a condition $p_2 \leq p_1, p_2 = \langle x^{p_2}, y^{p_2}, z^{p_2}, \mathfrak{m}^{p_2}\rangle$. By eventually strengthening p_2 if necessary, we can assume that $\mathfrak{m}^{p_2}$ is $M \cap H_\kappa$-full (L9.2) and there is an element x of A above p_2. We shall show that this element x is actually in $A \cap M$, which will finish the proof of Lemma 14. To do this, we first define a condition p_3, a kind of projection of p_2 to $P \cap M$. It is designed to capture all the content of p_2 understandable from within M. So, let $x^{p_3} = x^{p_2} \cap M, y^{p_3} = y^{p_2} \cap M$, for every γ in y^{p_3} we let $z^{p_3}(\gamma) = z^{p_2}(\gamma) \restriction x^{p_3}$ and set $\mathfrak{m}^{p_3} = pr_{M \cap H_\kappa}(\mathfrak{m}^{p_2})$.

CLAIM 16: $p_3 \in P \cap M, p_3 \geq p_2$.

Proof: Obviously, $p_3 \in P$ and as p_3 is a finite set of objects in M, we can conclude that $p_3 \in M$ as well. We must verify that $p_3 \geq p_2$. There are three clauses to the definition of $\geq$. (D10.4) is immediate. For (D10.5), let $\gamma \neq \delta$ be both in $y^{p_3} \subset y^{p_2}$ and let $\beta \in x^{p_2} \setminus x^{p_3}$. In particular, we have that both γ, δ are in $M \cap H_\kappa$ and $\beta \geq M \cap \omega_1$. Applying (D10.6) for the relation $p_2 \leq p_1$, we get that $z^{p_2}(\gamma)(\beta) \neq z^{p_2}(\delta)(\beta)$ and (D10.5) follows. (This is the main point in requiring (D10.6).) We still have to check (D10.6). There are two cases. First, if $\gamma \neq \delta$ are both in M and $\beta \in x^{p_2}$ then $z^{p_3}(\gamma)(\beta), z^{p_3}(\delta)(\beta)$ are undefined iff $\beta \geq M \cap \omega_1$ and in this case $z^{p_3}(\gamma)(\beta) \neq z^{p_3}(\delta)(\beta)$—this follows from (D10.6) for the relation $p_1 \geq p_2$. So in this case (D10.6) for γ, δ holds. The other case is that one of γ, δ is not in M, let us say that $\gamma \notin M$. But since for every $\alpha \in \mathrm{dom}(\mathfrak{m}^{p_3})$ and every $N \in \mathfrak{m}^{p_3}(\alpha)$ we have $N \in M$ and so $N \subset M$ and $\gamma \notin N$, we conclude that the third universal quantifier in (D10.6) is vacuous, and (D10.6) is vacuously true for this case. ∎

The point in the definition of the condition p_3 is that there are in some sense "many" extensions of p_3 in M which are still compatible with p_2:

CLAIM 17: *Let $q \in M \cap P, q \leq p_3$. If $(y^q \setminus y^{p_3}) \cap \bigcup\{N \in \mathfrak{m}^{p_2}(\alpha): \alpha \in M\} = 0$ then q is compatible with p_2.*

Proof: We shall produce a lower bound r of q and p_2. We set $x^r = x^q \cup x^{p_2}$ and $y^r = y^q \cup y^{p_2}$. The values of the function z^r with domain y^r are obtained as follows:

(1) $z^r(\gamma)$ extends $z^q(\gamma) \cup z^{p_2}(\gamma)$,
(2) at all ordinals $\alpha \in x^r$ where the values $z^r(\gamma)(\alpha)$ are not decided in (1), we fill in the values to be pairwise different integers different from integers used in q and p_2.

Step (2) can be easily done since only finitely many integers have been used in q and p_2. Finally, the matrix $\mathfrak{m}^r$ will be any $\mathfrak{M}$-lower bound of $\mathfrak{m}^q$ and $\mathfrak{m}^{p_2}$. Since $\mathfrak{m}^q \leq_{\mathfrak{M}} \mathfrak{m}^{p_3} = \mathrm{pr}_{M \cap H_\kappa} \mathfrak{m}^{p_2}$ and $\mathfrak{m}^q \in M \cap H_\kappa$, such a bound exists by (L9.3).

Well, now $r = \langle x^r, y^r, z^r, \mathfrak{m}^r \rangle$ is a condition in P. We should verify that $r \leq q, r \leq p_2$. The proof of (D10.4,5,6) for these inequalities breaks into numerous cases and subcases, which are quite simple and follow the lines of the proof of Claim 16. We show the only point in the argument where we use the special properties of q. This is when we check (D10.6) for the inequality $r \leq p_2$, and we have ordinals $\gamma \neq \delta$ both in M, ordinals $\alpha \leq \beta < M \cap \omega_1$ such that $\alpha \in \mathrm{dom}(\mathfrak{m}^{p_2})$ and $\beta \in x^r$, and a model $N \in \mathfrak{m}^{p_2}(\alpha)$. So we have that both γ, δ are in y^q and $\beta \in x^q$. There are two subcases:

(1) Both γ, δ are already in y^{p_3}. Here, either $\beta \in x^{p_3} \subset x^{p_2}$. Then the "if" clause is false and the whole formula inside (D10.6) is true. Or $\beta \in x^q \setminus x^{p_3}$. Then $z^r(\gamma)(\beta) = z^q(\gamma)(\beta) \neq z^q(\delta)(\beta) = z^r(\delta)(\beta)$ as required by (D10.6); this holds by (D10.5) for the inequality $q \leq p_3$ and the definition of z^r.
(2) One of γ, δ is not in y^{p_3}, let us say that $\gamma \in y^q \setminus y^{p_3}$. In this case, by our assumption on the condition q we have that necessarily $\gamma \notin N$. So (D10.6) is vacuously true in this case.

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The final step in the proof is to use the elementarity of M to find an extension p_4 of p_3 in M which is compatible with p_2 and already has an element of A above it. This is the main technical point in the whole argument.

Claim 18: *There is a condition $p_4 \in M \cap P, p_4 \leq p_3$, such that p_4 has an element of A above it and $(y^{p_4} \setminus y^{p_3}) \cap \bigcup\{N \in \mathfrak{m}^{p_2}(\alpha): \alpha \in M\} = 0$.*

Proof: Let $\gamma_0, \gamma_1, \dots \gamma_n$ be a list of all ordinals in $y^{p_2} \setminus y^{p_3}$. So $\gamma_j \in \kappa \setminus M$, for each $j \leq n$. Let $\delta_j = \min(M \cap \mathrm{Ord} \setminus \gamma_j)$. By the elementarity of M, each δ_j is a limit ordinal of uncountable cofinality. We list $\delta_j, j \leq n$ in an increasing order without repetitions as $\langle \epsilon_j : j \leq m \rangle$. By the elementarity of M once again, given ordinals $\zeta_j : j \leq m$ in M with $\zeta_j < \epsilon_j$, we have

$$(*) \quad M \models \exists q \leq p_3 \text{ with an element of } A \text{ above it and } y^q \setminus y^{p_3} \subset \bigcup_{j \leq m} [\zeta_j, \epsilon_j).$$

Here, $[\alpha, \beta)$ stands for the interval $\{\gamma : \alpha \leq \gamma < \beta\}$. $(*)$ holds because in H_λ, this sentence is witnessed by p_2. Notice the role of Claim 15 here! Within M, we shall produce many q's as in $(*)$. How many do we need?

Let $N_0 \in \mathfrak{m}^{p_2}(\alpha)$ for some $\alpha \in M$. Even though not necessarily $N_0 \in M$, we still can capture some important properties of N_0 in M. This is because N_0 is isomorphic to some $N_1 \in \mathfrak{m}^{p_2}(\alpha) \cap M$, in particular $N_0 \cap \kappa$ and $N_1 \cap \kappa$ have the same ordertype, which is less than $M \cap \omega_1$. It follows that there is an indecomposable ordinal $\beta \in M \cap \omega_1$ such that for every $\alpha \in \text{dom}(\mathfrak{m}^{p_2}) \cap M$ and every $N \in \mathfrak{m}^{p_2}(\alpha)$ we have that $N \cap \kappa$ has ordertype less than β. This gives a rough answer to the previous question: we shall produce "β many q's as in $(*)$".

Work in M. By induction on $\nu < \beta$ we define $\zeta_j^\nu, j \leq m$ and y_ν, q_ν so that:

(1) $\zeta_0^0 = 0$ and $\zeta_{j+1}^0 = \epsilon_j$ for $j < m$; moreover $\mu < \nu < \beta$ implies $\zeta_j^\mu < \zeta_j^\nu < \epsilon_j$,

(2) q_ν is the $\ll$-least element of P witnessing $(*)$ for $\zeta_j = \zeta_j^\nu$; and $y_\nu = y^{q_\nu} \setminus y^{p_3}$,

(3) $\zeta_j^{\nu+1} = \max(y_\nu \cap \epsilon_j) + 1$; for limit ν we set $\zeta_j^\nu = \bigcup_{\mu<\nu} \zeta_j^\mu$.

All of this is possible since (a) from $(*)$ we can be sure that at each $\nu < \beta$ the desired q_ν as in (2) exists and (b) if $\nu < \beta$ is limit then $\zeta_j^\nu = \bigcup_{\mu<\nu} \zeta_j^\mu < \epsilon_j$ as $\text{cof}(\epsilon_j) > \omega$ and ν is countable.

SUBCLAIM 19: *There is $\nu < \beta$ such that $y_\nu \cap \bigcup \{N \in \mathfrak{m}^{p_2}(\alpha) : \alpha \in M\} = 0$.*

Proof of the Subclaim: Notice that by our construction, y_ν's are pairwise disjoint. Assume for contradiction that for each $\nu < \beta$ there are $\alpha \in M$ and $N \in \mathfrak{m}^{p_2}(\alpha)$ such that $y_\nu \cap N \neq 0$. By the indecomposability of β, there are a set $B \subset \beta$ of ordertype β, integer $j \leq m$ and a model $N \in \mathfrak{m}^{p_2}(\alpha)$ such that for each $\nu \in B$, we have $y_\nu \cap N \cap [\epsilon_{j-1}, \epsilon_j) \neq 0$, where ϵ_{-1} is understood to be 0. But then necessarily o.t.$(N \cap \epsilon_j) \geq$ o.t.$B = \beta$, contradicting our choice of β. ∎

Thus if ν is as in the Subclaim, the condition $p_4 = q_\nu$ witnesses the statement of Claim 18. ∎

Now the conditions p_2 and p_4 are compatible and both have an element of A above them. Since $A \subset P$ is an antichain, we conclude that these two elements of A are identical. But as $p_4 \in M$, the unique element of A above p_4 is in M as well and we are through. ∎

The argument for Theorem 3 is now routine. Fix a generic filter $G \subset P$ and in the generic extension, for each $\gamma \in \kappa$, set $f_\gamma = \bigcup\{z^p(\gamma): p \in G \text{ and } \gamma \in \mathrm{dom}(p)\}$.

LEMMA 20: *In $V[G]$, the family $\{f_\gamma: \gamma \in \kappa\} \subset {}^{\omega_1}\omega$ is strongly almost disjoint.*

Proof: This is an exercise in identifying the required dense subsets of P. ∎

By Lemma 11, the poset P preserves cardinals $\geq \aleph_2$, by Lemma 14 $\aleph_1$ is preserved and Lemma 20 shows that P adds a strongly almost disjoint family of size κ. Theorem 3 has been proven. ∎

3. Concluding remarks

We should note that there is a different proof of Theorem 3 which is perhaps more traditional in the sense that it uses an iteration of σ-closed and c.c.c. forcings. This method allows a generalization to arbitrary ω_{n+1}, ω_n to obtain

THEOREM 21: *Let GCH hold, let n be an arbitrary integer and κ an arbitrary cardinal. Then there is a cardinal-preserving forcing notion P such that $P \Vdash$ "there is a family F of size κ consisting of functions from ω_{n+1} to ω_n such that for any two distinct $f, g \in F$ the set $\{\alpha < \omega_{n+1}: f(\alpha) = g(\alpha)\}$ is finite".*

The simplest open question therefore is

QUESTION 22: *Is it possible to prove such a theorem for $\omega_{\omega+1}, \omega_\omega$?*

References

[B] J. E. Baumgartner, *Almost-disjoint sets, the dense set problem and the partition calculus*, Annals of Mathematical Logic **9** (1976), 401–439.

[J] T. J. Jech, *Set Theory*, Academic Press, New York, 1978.

[JS] T. J. Jech and S. Shelah, *A note on canonical functions*, Israel Journal of Mathematics **68** (1989), 376–380.

[S] S. Shelah, *Proper Forcing*, Lecture Notes in Mathematics **940**, Springer-Verlag, Berlin, 1982.

[T1] S. Todorcevic, *Partition problems in topology*, Contemporary Mathematics **84**, American Mathematical Society, Providence, R.I., 1989.

[T2] S. Todorcevic, *Remarks on Martin's Axiom and the continuum hypothesis,* Canadian Journal of Mathematics **43** (1991), 832–851.

[Z] J. Zapletal, *Some results in set theory and Boolean algebras,* Ph.D. thesis, The Pennsylvania State University, 1995.